BERNOULLI MEASURES

In this note, we use Caratheodory's Extension Theorem to define Bernoulli measures. These measures are of fundamental importance in probability theory: they are used to model the outcomes of an infinite sequence of independent identically distributed discrete random variables, such as an infinite sequence of (possibly biased) independent coin tosses.

Let $X = \{0,1\}^{\mathbb{N}}$, equipped with the product topology τ. Elements of X are infinite sequences $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$, where $x_i = 0$ or $x_i = 1$ (which we will think of as a realization of our stochastic process).

We let $\pi_i \colon X \to \{0,1\}$ denote the projection map on the i-th coordinate, namely $\pi_i(\mathbf{x}) = x_i$. We recall that, by definition, τ is *the coarsest topology that makes the projection maps π_i continuous*. In particular, any *cylinder set C*, that is, any set of the form

$$C = \bigcap_{i\in S} \pi_i^{-1}(x_i), \qquad \text{for any } \textit{finite } S \subset \mathbb{N} \text{ and } x_i \in \{0,1\}$$

is an open set. For example, the set of all sequences that start with 0 is a cylinder set $\{\mathbf{x} : x_1 = 0\} = \pi_1^{-1}(0)$.

We recall the following fact from basic topology.

Proposition 1. *The topological space (X,τ) is second-countable, namely there exists a countable basis of τ consisting of cylinders.*

Let

$$\mathscr{A}_0 := \{\text{finite union of cylinders}\}.$$

We notice that $\mathscr{A}_0$ is an algebra of sets: finite intersections of cylinders is again a cylinder and the complement of a cylinder is a finite union of cylinders.

Let us now fix $p = p_0 \in [0,1]$ and let $p_1 = 1 - p \in [0,1]$. We define a function $\ell \colon \mathscr{A}_0 \to [0,1]$ as follows. If $S \subset \mathbb{N}$ is a finite set and $C = \cap_{i\in S}\pi_i^{-1}(x_i)$, where $x_i \in \{0,1\}$, is a cylinder, then we let

$$\ell(C) = \prod_{i\in S} p_{x_i}.$$

For example, if C is the cylinder consisting of sequences $\mathbf{x}$ such that $x_2 = 0$, $x_3 = 1$, and $x_5 = 0$, namely

$$C = \pi_2^{-1}(0) \cap \pi_3^{-1}(1) \cap \pi_5^{-1}(0),$$

then

$$\ell(C) = p_0 \cdot p_1 \cdot p_0 = p^2(1-p).$$

We think of $\ell(C)$ as the probability of the event C.

If C_1 and C_2 are disjoint cylinders, then we set $\ell(C_1 \cup C_2) = \ell(C_1) + \ell(C_2)$. We can now extend ℓ to all of $\mathscr{A}_0$, since any finite union of cylinders can be also written as a finite *disjoint* union of cylinders.

We leave the following fact as an exercise.

Exercise 2. *The function ℓ is a premeasure on $\mathscr{A}_0$.* (Hint: *it might be useful to note that X is compact and cylinders are clopen sets).*

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By Caratheodory's Extension Theorem, ℓ extends in a unique way to a measure μ on $\sigma(\mathscr{A}_0)$, which is called a *Bernoulli measure*. Notice that, by Proposition 1, the sigma-algebra $\sigma(\mathscr{A}_0)$ coincides with the Borel sigma-algebra $\mathscr{B} = \sigma(\tau)$.

In the probabilistic interpretation, the Bernoulli measure $\mu(A)$ of a Borel set A is the probability of the event A. For example, let

$$A := \{\mathbf{x} \in X \ : \ x_i = 0 \text{ for finitely many } i\text{'s } \}.$$

If we let

$$C_{m,k} := \bigcap_{j=0}^{k-1} \pi_{m+j}^{-1}(1) = \{\mathbf{x} \in X \ : \ x_m = 1, x_{m+1} = 1, \dots, x_{m+k-1} = 1\},$$

then

$$A = \bigcup_{m\geq 1} \bigcap_{k\geq 1} C_{m,k},$$

which proves that $A \in \sigma(\mathscr{A}_0) = \mathscr{B}$.

Assume that $p > 0$. By the properties of measures, we have

$$\mu(A) \leq \sum_{m=1}^{\infty} \mu\left(\bigcap_{k\geq 1} C_{m,k}\right) = \sum_{m=1}^{\infty} \lim_{k\to\infty} \mu\left(C_{m,k}\right) = \sum_{m=1}^{\infty} \lim_{k\to\infty} (1-p)^k = \sum_{m=1}^{\infty} 0 = 0,$$

which means that with probability 1 there will be infinitely many occurrences of the symbol 0.

Exercise 3. *Generalize the construction of Bernoulli measures to $X = Y^{\mathbb{N}}$ and $X = Y^{\mathbb{Z}}$, where Y is any finite set.*